Presentations of Factorizable Inverse Monoids

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Abstract

It is well-known that an inverse monoid is factorizable if and only if it is a homomorphic image of a semidirect product of a semilattice (with identity) by a group. We use this structure to describe a presentation of an arbitrary factorizable inverse monoid in terms of presentations of its group of units and semilattice of idempotents, together with some other data. We apply this theory to quickly deduce a well known presentation of the symmetric inverse monoid on a finite set.

Keywords: Factorizable inverse monoid, presentations, symmetric inverse monoid.

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1 Introduction

A semigroup M is said to be *factorizable* if $M = EG$ where E is a set of idempotents of M, and G is a subgroup of M. The study of factorizable *inverse* semigroups was initiated in [1]. In this context, M has an identity which is the identity of G. Furthermore, $E = E(M)$ is the semilattice of idempotents of M, and $G = G(M)$ is the group of units of M. The results concerning factorizable inverse monoids (henceforth FIMs) are especially nice; for example, the symmetric inverse monoid $\mathcal{I}_X$ on a set X is factorizable if and only if X is finite [1]. There are many other examples of FIMs – see for example [4, 5, 6, 8, 9, 10, 14]. Further studies of factorizable semigroups, both inverse and otherwise, have been conducted in [3, 12, 17, 22, 23].

It was shown in [12] that an inverse monoid is factorizable if and only if it is a homomorphic image of a semidirect product of a semilattice (with identity) by a group. We review this theory in Section 2 and then use it to describe presentations of FIMs in Section 3. In Section 4 we apply our results to deduce the presentation of $\mathcal{I}_X$ when $X = \{1, \ldots, n\}$ originally due to Popova [19]. For other proofs of Popova's presentation, see [5] and [13]. For a useful alternative presentation of $\mathcal{I}_X$ in the context of Hecke algebras, see [20, 21]. The method we use in the final section mirrors the approach of Wilkinson [24], who gives his own proof of the celebrated McAlister P-Theorem ([12] Ch. 7). Wilkinson uses semilattices of subsets under intersection, though we find it more convenient to use union.

2 The Structure of Factorizable Inverse Monoids

The results of this section are well-known (see [12], pp204–206). Proofs of certain results are included here for the convenience of the reader.

Suppose that G is a group and that E is a semilattice (a monoid of commuting idempotents). Without causing confusion, we denote the identities of both E and G by 1. Suppose also that for each $g \in G$ we have an automorphism $\varphi_g : E \to E : e \mapsto e^g$ such that the map $\varphi : G \to \mathrm{Aut}(E) : g \mapsto \varphi_g$ is an antihomomorphism. We may then form the semidirect product $E \rtimes G = E \rtimes_\varphi G = \big\{(e, g) \,\big|\, e \in E,\ g \in G\big\}$ with multiplication defined by

$$(e_1, g_1)(e_2, g_2) = (e_1 e_2^{g_1}, g_1 g_2).$$

Let $(1, G) = \big\{(1, g) \,\big|\, g \in G\big\}$ and $(E, 1) = \big\{(e, 1) \,\big|\, e \in E\big\}$. It is easy to verify the following.

Lemma 1 *The monoid $E \rtimes G$ is a factorizable inverse monoid with semilattice of idempotents $(E, 1) \cong E$ and group of units $(1, G) \cong G$.* $\square$

Suppose that for each $e \in E$, there is a subgroup $G_e \leq G$ such that $G_1 = \{1\}$ and

$$gG_eg^{-1} = G_{e^g} \qquad \forall e \in E, g \in G \qquad (G_e1)$$
$$G_e \vee G_f \subseteq G_{ef} \qquad \forall e, f \in E \qquad (G_e2)$$
$$e^g = e \qquad \forall e \in E, g \in G_e. \qquad (G_e3)$$

Here we have used the notation $H \vee H' = \langle H \cup H' \rangle$ for the join of two subgroups H and H' of G. Define an equivalence $\sim$ on $E \rtimes G$ by

$$(e_1, g_1) \sim (e_2, g_2) \quad \text{if and only if} \quad e_1 = e_2 \quad \text{and} \quad g_1 g_2^{-1} \in G_{e_1}.$$

Lemma 2 *The equivalence $\sim$ is a congruence.*

Proof Suppose that $(e_1, g_1) \sim (f_1, h_1)$ and $(e_2, g_2) \sim (f_2, h_2)$. Then $e_1 = f_1$, $e_2 = f_2$, and $g_1 h_1^{-1} \in G_{e_1}$, $g_2 h_2^{-1} \in G_{e_2}$. But then

$$\begin{aligned}
g_1 g_2 (h_1 h_2)^{-1} &= (g_1 h_1^{-1}) h_1 (g_2 h_2^{-1}) h_1^{-1} \\
&\in G_{e_1} h_1 G_{e_2} h_1^{-1} \\
&= G_{e_1} G_{e_2^{h_1}} && \text{by } (G_e1) \\
&\subseteq G_{e_1} \vee G_{e_2^{h_1}} \\
&\subseteq G_{e_1 e_2^{h_1}} && \text{by } (G_e2).
\end{aligned}$$

Also, since $g_1h_1^{-1} \in G_{e_1} \subseteq G_{e_1} \vee G_{e_2^{h_1}} \subseteq G_{e_1e_2^{h_1}}$, we have

$$\begin{aligned} e_1e_2^{h_1} &= (e_1e_2^{h_1})^{g_1h_1^{-1}} && \text{by } (G_e3) \\ &= e_1^{g_1h_1^{-1}}(e_2^{h_1})^{g_1h_1^{-1}} \\ &= e_1e_2^{(g_1h_1^{-1})h_1} \\ &= e_1e_2^{g_1}, \end{aligned}$$

so that $(e_1,g_1)(e_2,g_2) \sim (f_1,h_1)(f_2,h_2)$. □

We now define $\widetilde{G} = (E \rtimes G)/{\sim}$. For $e \in E, g \in G$, let $[e,g]$ denote the $\sim$-class of (e,g) in $E \rtimes G$. Also write $[1,G] = \big\{[1,g] \,\big|\, g \in G\big\}$ and $[E,1] = \big\{[e,1] \,\big|\, e \in E\big\}$. The proof of the following is straightforward.

Proposition 3 *The natural map $(e,g) \mapsto [e,g]$ is injective on $(1,G)$ and $(E,1)$. Thus $\widetilde{G}$ is a factorizable inverse monoid with group of units $[1,G] \cong G$ and semilattice of idempotents $[E,1] \cong E$.* □

Proposition 4 *Let M be a factorizable inverse monoid with group of units G and semilattice of idempotents E. Then $M \cong \widetilde{G}$ arises from the construction above.*

Proof For $e \in E$ and $g \in G$ we define $e^g = geg^{-1}$. The maps $\varphi_g : e \mapsto e^g$ are automorphisms of E, and $\varphi : G \to \mathrm{Aut}(E) : g \mapsto \varphi_g$ is clearly an antihomomorphism. Thus we may form $E \rtimes G$ as above. For $e \in E$ let $G_e = \{g \in G \,|\, eg = e\}$. It is routine to check that the G_e are subgroups of G, and that they satisfy conditions (G_e1)—(G_e3). In particular we may form $\widetilde{G} = (E \rtimes G)/{\sim}$. It finally remains to observe that $eg \mapsto [e,g]$ is a well defined isomorphism $M \to \widetilde{G}$. □

Remark 5 Phrased differently, (G_e2) states that $\Gamma : e \mapsto G_e$ is an (anti-)representation of E as a *poset* in $\mathrm{Sub}(G)$. Conditions (G_e1) and (G_e3) link Γ with the (anti-)representation $\varphi : G \to \mathrm{Aut}(E)$. The role of the subgroups G_e is to provide the *kernel normal system* (see [2], p60) for the congruence $\sim$, which consists of the subsemigroups $(e,G_e) = \big\{(e,g) \,\big|\, g \in G_e\big\} \subseteq E \rtimes G$. We see a prototype of the McAlister theorem here, as $E \rtimes G$ is E-unitary and $\sim$ is idempotent-separating.

3 Presentations of Factorizable Inverse Monoids

In this section we make use of Propositions 3 and 4 to describe a presentation of an arbitrary FIM M. The necessary ingredients are presentations of $E = E(M)$ and $G = G(M)$, information about the (anti-)action of G on E, and generating sets for the subgroups G_e.

First we establish the notation we will be using throughout. Let X be an alphabet (a set whose elements are called letters), and denote by X^* the free monoid on X. For

$R \subseteq X^* \times X^*$, let $R^\sharp$ denote the smallest congruence on X^* containing R. We say that a monoid M has (monoid) presentation $\langle X \,|\, R \rangle$ if $M \cong X^*/R^\sharp$. An element $(w_1, w_2) \in R$ is called a relation, and is often written as $w_1 = w_2$. All presentations we consider will be monoid presentations.

Suppose now that M is an arbitrary FIM. Then by Proposition 4, we may identify M with $\widetilde{G} = (E \rtimes G)/\sim$ using the notation of Section 2. Suppose that E and G have presentations $\langle X_E \,|\, R_E \rangle$ and $\langle X_G \,|\, R_G \rangle$ respectively, so there exist monoid epimorphisms $\alpha : X_E^* \to E$ and $\beta : X_G^* \to G$ such that $\ker \alpha = R_E^\sharp$ and $\ker \beta = R_G^\sharp$. For each $e \in E$, choose $\hat{e} \in e\alpha^{-1}$ and for each $g \in G$, choose $\hat{g} \in g\beta^{-1}$. We may assume that these choices are made so that $\widehat{x\alpha} = x$ and $\widehat{y\beta} = y$ for each $x \in X_E$ and $y \in X_G$. Put

$$R_\rtimes = \left\{ (yx, \widehat{x\alpha^{y\beta}} y) \,\middle|\, x \in X_E\,,\ y \in X_G \right\}.$$

It is well known (see for example [11]) that $E \rtimes G$ has presentation

$$\langle X_G \sqcup X_E \,|\, R_G \sqcup R_E \sqcup R_\rtimes \rangle.$$

Suppose now that for each $e \in E$ we have a subset $\Sigma_e \subseteq G$ such that G_e is generated as a submonoid by Σ_e. (We could take $\Sigma_e = G_e$, but in applications we would choose Σ_e minimally to avoid superfluous relations.) Put

$$R_\sim = \left\{ (\hat{e}\hat{g}, \hat{e}) \,\middle|\, e \in E\,,\ g \in \Sigma_e \right\}.$$

Theorem 6 *The factorizable inverse monoid $M \cong (E \rtimes G)/\sim$ has presentation*

$$\langle X_G \sqcup X_E \,|\, R_G \sqcup R_E \sqcup R_\rtimes \sqcup R_\sim \rangle.$$

Proof Put $\approx\, = (R_G \sqcup R_E \sqcup R_\rtimes \sqcup R_\sim)^\sharp$ and define $\phi : (X_G \sqcup X_E)^* \to (E \rtimes G)/\sim$ by

$$x\phi = [x\alpha, 1] \text{ and } y\phi = [1, y\beta] \qquad \text{for } x \in X_E,\ y \in X_G.$$

Then ϕ is surjective since α and β are surjective and $(E \rtimes G)/\sim$ is factorizable, so it remains to show that $\ker \phi = \,\approx$. Now $\approx\, \subseteq \ker \phi$ since the relations hold as equations in $(E \rtimes G)/\sim$ after substituting the images of generators. Suppose $w_1, w_2 \in (X_G \sqcup X_E)^*$ and $w_1\phi = w_2\phi$. Using $R_\rtimes$, $w_i \approx \hat{e}_i\hat{g}_i$ $(i = 1, 2)$ for some $e_i \in E$, $g_i \in G$. But then

$$[e_1, g_1] = w_1\phi = w_2\phi = [e_2, g_2],$$

so that $e_1 = e_2$ and $g_1 g_2^{-1} \in G_{e_1}$. Thus $g_1 g_2^{-1} = h_1 \cdots h_k$ for some $h_1, \ldots, h_k \in \Sigma_{e_1}$ and

$$\begin{aligned} w_1 &\approx \hat{e}_1\hat{g}_1 \approx \hat{e}_1\widehat{g_1 g_2^{-1}}\hat{g}_2 && \text{by } R_G \\ &\approx \hat{e}_1\hat{h}_1 \cdots \hat{h}_k\hat{g}_2 && \text{by } R_G \\ &\approx \hat{e}_1\hat{g}_2 && \text{by } R_\sim \\ &\approx \hat{e}_2\hat{g}_2 && \text{by } R_E. \end{aligned}$$

This completes the proof. $\square$

We complete this section by proving that if the subgroups G_e satisfy the stronger condition

$$G_e \vee G_f = G_{ef} \qquad\qquad \forall e,f \in E \qquad\qquad (G_e2)'$$

then the set $R_\sim$ defined above may be replaced by

$$R'_\sim = \big\{(x\hat{g}, x) \,\big|\, x \in X_E\,,\ g \in \Sigma_{x\alpha}\big\}.$$

Theorem 7 *Suppose that the factorizable inverse monoid $M \cong (E \rtimes G)/\!\sim$ satisfies condition $(G_e2)'$. Then M has presentation*

$$\langle X_G \sqcup X_E \,|\, R_G \sqcup R_E \sqcup R_\rtimes \sqcup R'_\sim \rangle.$$

Proof Put $\approx' = (R_G \sqcup R_E \sqcup R_\rtimes \sqcup R'_\sim)^\sharp$. Since $R'_\sim \subseteq R_\sim$, it suffices, by the previous theorem, to show that $R_\sim \subseteq \approx'$. Let $e \in E$ and $g \in G_e$. We must prove that $\hat{e}\hat{g} \approx' \hat{e}$. Now $e = (x_1\alpha)\cdots(x_k\alpha)$ for some $x_1,\ldots,x_k \in X_E$. By $(G_e2)'$, we have $G_e = G_{x_1\alpha} \vee \cdots \vee G_{x_k\alpha}$, so $g = g_1\cdots g_\ell$ for some $g_1,\ldots,g_\ell \in G_{x_1\alpha} \cup \cdots \cup G_{x_k\alpha}$. For each $i \in \{1,\ldots,\ell\}$, there exists $m_i \in \{1,\ldots,k\}$ such that $g_i \in G_{x_{m_i}\alpha}$, and so $g_i = h_{i1}\cdots h_{in_i}$ for some $h_{i1},\ldots,h_{in_i} \in \Sigma_{x_{m_i}\alpha}$. But then

$$\begin{aligned}
\hat{e}\hat{g} &\approx' x_1\cdots x_k(\hat{h}_{11}\cdots\hat{h}_{1n_1})\cdots(\hat{h}_{\ell 1}\cdots\hat{h}_{\ell n_\ell}) && \text{by } R_E \text{ and } R_G\\
&\approx' x_1\cdots x_k x_{m_1}(\hat{h}_{11}\cdots\hat{h}_{1n_1})\cdots(\hat{h}_{\ell 1}\cdots\hat{h}_{\ell n_\ell}) && \text{by } R_E\\
&\approx' x_1\cdots x_k x_{m_1}(\hat{h}_{21}\cdots\hat{h}_{2n_2})\cdots(\hat{h}_{\ell 1}\cdots\hat{h}_{\ell n_\ell}) && \text{by } R'_\sim\\
&\approx' x_1\cdots x_k(\hat{h}_{21}\cdots\hat{h}_{2n_2})\cdots(\hat{h}_{\ell 1}\cdots\hat{h}_{\ell n_\ell}) && \text{by } R_E\\
&\approx' x_1\cdots x_k && \text{by a simple induction}\\
&\approx' \hat{e} && \text{by } R_E.
\end{aligned}$$

$\square$

Remark 8 Condition $(G_e2)'$ is a sufficient condition for M to embed in the coset monoid of G (see [4]), and is also necessary if M is finite (see [7]). Although many FIMs satisfy this condition (see for example [4, 8]), certainly not all do. The example we consider in Section 4 does not. Finally we note that Theorem 7 holds if condition $(G_e2)'$ is replaced by

$$(\forall e \in E)\ \ (\exists x_1,\ldots,x_k \in X_E) \qquad e = (x_1\alpha)\cdots(x_k\alpha) \ \text{ and } \ G_e = G_{x_1\alpha} \vee \cdots \vee G_{x_k\alpha},$$

or the even weaker condition

$$G_e = \bigvee_{\substack{x \in X_E\\ e(x\alpha)=e}} G_{x\alpha} \qquad\qquad (\forall e \in E).$$

Monoids satisfying these conditions occur naturally when braid equivalence is modified (see [6]).

4 The Symmetric Inverse Monoid

We conclude by using Theorem 6 to obtain a well-known presentation of $\mathcal{I}_n$, the symmetric inverse monoid on the set $\mathbf{n} = \{1, \ldots, n\}$. Let $\mathcal{P} = \mathcal{P}_n = \{A \,|\, A \subseteq \mathbf{n}\}$ be the power set of $\mathbf{n}$ which is a semilattice under $\cup$ with identity $\emptyset$, and let $\mathcal{S} = \mathcal{S}_n$ be the symmetric group on $\mathbf{n}$. For $A \in \mathcal{P}$ and $\pi \in \mathcal{S}$, define

$$A^\pi = \{a\pi^{-1} \,|\, a \in A\}.$$

Then for each $\pi \in \mathcal{S}$, $\varphi_\pi : A \mapsto A^\pi$ defines an automorphism of $\mathcal{P}$, and the map $\varphi : \pi \mapsto \varphi_\pi$ is an antihomomorphism $\mathcal{S} \to \mathrm{Aut}(\mathcal{P})$. Thus we may form $\mathcal{P} \rtimes \mathcal{S}$ as above. For $A \in \mathcal{P}$ let $A^c = \mathbf{n} \setminus A$, and put

$$\mathcal{S}_A = \big\{\pi \in \mathcal{S} \,\big|\, i\pi = i \ \ (\forall i \in A^c)\big\}.$$

One may easily check that these subgroups satisfy $\mathcal{S}_\emptyset = \{1\}$ and

$$\begin{aligned} \pi\mathcal{S}_A\pi^{-1} &= \mathcal{S}_{A^\pi} && \forall A \in \mathcal{P}, \pi \in \mathcal{S} \\ \mathcal{S}_A \vee \mathcal{S}_B &\subseteq \mathcal{S}_{A\cup B} && \forall A, B \in \mathcal{P} \\ A^\pi &= A && \forall A \in \mathcal{P}, \pi \in \mathcal{S}_A. \end{aligned}$$

Thus, by Lemma 2, the equivalence $\sim$ on $\mathcal{P} \rtimes \mathcal{S}$ defined by

$$(A, \pi) \sim (B, \tau) \quad \text{if and only if} \quad A = B \quad \text{and} \quad \pi\tau^{-1} \in \mathcal{S}_A$$

is a congruence, and so we may form the quotient $(\mathcal{P} \rtimes \mathcal{S})/\!\sim$. Denote the $\sim$-class of $(A, \pi) \in \mathcal{P} \rtimes \mathcal{S}$ by $[A, \pi]$. The proof of the following is straightforward.

Lemma 9 *The map $\theta : [A, \pi] \mapsto \pi|_{A^c}$ defines an isomorphism from $(\mathcal{P} \rtimes \mathcal{S})/\!\sim$ to $\mathcal{I}_n$.* □

We now collect the relevant data needed to apply Theorem 6. It is well-known that $\mathcal{P}$ under either union (as in our case) or intersection (see for example [15], p115) is a free semilattice on n generators. Thus we have the following.

Proposition 10 *The power set $\mathcal{P}$ has presentation $\langle X_\mathcal{P} \,|\, R_\mathcal{P}\rangle$ where $X_\mathcal{P} = \{\varepsilon_1, \ldots, \varepsilon_n\}$ and $R_\mathcal{P}$ is the set of relations*

$$\begin{aligned} \varepsilon_i^2 &= \varepsilon_i && \text{for all } i && \text{(P1)} \\ \varepsilon_i\varepsilon_j &= \varepsilon_j\varepsilon_i && \text{for all } i, j. && \text{(P2)} \end{aligned}$$

□

Here $\alpha : X_\mathcal{P}^* \to \mathcal{P}$ is the epimorphism defined by $\varepsilon_i\alpha = \{i\}$ for each i.

Theorem 11 (Moore [18]) *The symmetric group $\mathcal{S}$ has presentation $\langle X_\mathcal{S} \,|\, R_\mathcal{S}\rangle$ where $X_\mathcal{S} = \{s_1, \ldots, s_{n-1}\}$ and $R_\mathcal{S}$ is the set of relations*

$$\begin{aligned} s_i^2 &= 1 && \text{for all } i && \text{(S1)} \\ s_is_j &= s_js_i && \text{if } |i-j| > 1 && \text{(S2)} \\ s_is_js_i &= s_js_is_j && \text{if } |i-j| = 1. && \text{(S3)} \end{aligned}$$

□

Here $\beta : X_{\mathcal{S}}^* \to \mathcal{S}$ is the epimorphism defined by $s_i\beta = (i, i+1)$ for each i. Now the set $R_{\rtimes}$ consists of the relations

$$s_i\varepsilon_j = \varepsilon_j s_i \qquad \text{if } j \neq i, i+1 \tag{$\rtimes$1}$$

$$s_i\varepsilon_i = \varepsilon_{i+1} s_i \tag{$\rtimes$2}$$

$$s_i\varepsilon_{i+1} = \varepsilon_i s_i. \tag{$\rtimes$3}$$

For each $A \in \mathcal{P}$ put $\Sigma_A = \big\{(i,j) \,\big|\, i,j \in A\,,\ i < j\big\}$. The following lemma is immediate from the definition of the subgroups $\mathcal{S}_A$.

Lemma 12 *Let $A \in \mathcal{P}$. Then $\mathcal{S}_A$ is generated by Σ_A.* □

For each $A \in \mathcal{P}$ choose $\varepsilon_A \in X_{\mathcal{P}}^*$ such that $\varepsilon_A\alpha = A \in \mathcal{P}$, and put $\widehat{A} = \varepsilon_A$. For $1 \leq i < j \leq n$, choose $t_{ij} \in X_{\mathcal{S}}^*$ such that $t_{ij}\beta = (i,j) \in \mathcal{S}$, and put $\widehat{(i,j)} = t_{ij}$. Here for example we could have

$$t_{ij} = (s_i \cdots s_{j-2})s_{j-1}(s_{j-2} \cdots s_i)$$
$$\varepsilon_A = \prod_{i \in A} \varepsilon_i.$$

Thus $R_{\sim}$ consists of the relations

$$\varepsilon_A t_{ij} = \varepsilon_A \qquad \text{if } A \in \mathcal{P}, \text{ and } i,j \in A. \tag{$\sim$}$$

The following is a consequence of Theorem 6.

Lemma 13 *The symmetric inverse monoid $\mathcal{I}_n$ has presentation*

$$\langle X_{\mathcal{S}} \sqcup X_{\mathcal{P}} \mid R_{\mathcal{S}} \sqcup R_{\mathcal{P}} \sqcup R_{\rtimes} \sqcup R_{\sim} \rangle. \qquad \square$$

We now show how this presentation may be reduced to the more familiar presentation of $\mathcal{I}_n$ due to Popova [19] (see also [13, 16] and references therein). As a first step we remove a number of the generators. With this in mind, let $e = \varepsilon_n$. By ($\rtimes$3) and (S1) we see that for any $i \in \mathbf{n}$ we have the relation

$$\varepsilon_i = (s_i \cdots s_{n-1})e(s_{n-1} \cdots s_i). \tag{$*$}$$

So we remove the generators ε_i, replacing their every occurrence in the relations by the word on the right hand side of ($*$), which we denote by e_i (notice in particular, that $e_n = e$). We denote the resulting relations by (P1)$'$, (P2)$'$, ($\rtimes$1)$'$, etc. Some additional relations which hold among the remaining generators are

$$e^2 = e \tag{I1}$$

$$es_i = s_i e \qquad \text{if } i \neq n-1 \tag{I2}$$

$$s_{n-1}es_{n-1}e = es_{n-1}es_{n-1} = es_{n-1}e. \tag{I3}$$

Indeed (I1) is part of (P1), (I2) is part of ($\rtimes$1), and (I3) follows from ($*$), (P2), ($\sim$), and (S1). Denote by R_I the set of relations (I1)—(I3), and let $\approx$ be the congruence on $\big(X_{\mathcal{S}} \sqcup \{e\}\big)^*$ generated by $R_{\mathcal{S}} \sqcup R_I$.

For a word $w = s_{i_1} \cdots s_{i_k} \in X_{\mathcal{S}}^*$, denote by w^{-1} the word $s_{i_k} \cdots s_{i_1}$ so that by (S1) we have $ww^{-1} \approx w^{-1}w \approx 1$. For $i \in \mathbf{n}$, let $c_i = s_{n-1} \cdots s_i \in X_{\mathcal{S}}^*$, so that $e_i = c_i^{-1} e c_i$.

Lemma 14 *If $i, j \in \mathbf{n}$ and $i < n$, then*

$$c_j s_i \approx \begin{cases} s_i c_j & \text{if } \ i < j-1 \\ c_{j-1} & \text{if } \ i = j-1 \\ c_{j+1} & \text{if } \ i = j \\ s_{i-1} c_j & \text{if } \ i > j. \end{cases}$$

Proof The first case follows immediately from (S2), the second by definition, the third from (S1), and the fourth from (S2) and (S3). □

Corollary 15 *If $i, j \in \mathbf{n}$ and $i < n$ then*

$$s_i e_j s_i \approx \begin{cases} e_j & \text{if } \ j \neq i, i+1 \\ e_{i+1} & \text{if } \ j = i \\ e_i & \text{if } \ j = i+1. \end{cases}$$

Proof This follows quickly from Lemma 14, relation (I2), and the fact that $e_j = c_j^{-1} e c_j$. □

Notice in particular that $s_i e_j s_i \approx e_{j(s_i\beta)}$. By induction we have the following.

Corollary 16 *Let $w \in X_{\mathcal{S}}^*$ and $i \in \mathbf{n}$. Then $w^{-1} e_i w \approx e_{i(w\beta)}$.* □

Theorem 17 (Popova [19]) *The symmetric inverse monoid $\mathcal{I}_n$ has presentation*

$$\langle X_{\mathcal{S}} \sqcup \{e\} \mid R_{\mathcal{S}} \sqcup R_I \rangle.$$

Proof All that remains is to show that relations (P1)$'$—(P2)$'$, ($\rtimes$1)$'$—($\rtimes$3)$'$, and ($\sim$)$'$ are implied by $R_{\mathcal{S}} \sqcup R_I$. Now (P1)$'$ easily follows from (I1) and (S1). For (P2)$'$, suppose that $i, j \in \mathbf{n}$. Choose $\pi \in \mathcal{S}$ such that $i = (n-1)\pi$ and $j = n\pi$, and let $w \in \pi\beta^{-1}$. Then by (S1), (I3), and Corollary 16, we have

$$\begin{aligned} e_i e_j = e_{(n-1)\pi} e_{n\pi} &\approx w^{-1} e_{n-1} w w^{-1} e_n w \approx w^{-1} e_{n-1} e_n w \\ &= w^{-1} s_{n-1} e s_{n-1} e w \approx w^{-1} e s_{n-1} e s_{n-1} w = w^{-1} e_n e_{n-1} w \approx e_j e_i. \end{aligned}$$

Relations ($\rtimes$1)$'$—($\rtimes$3)$'$ follow from $R_{\mathcal{S}} \sqcup R_I$ by Corollary 15 and (S1). If $A = \{i_1, \ldots, i_k\} \in \mathcal{P}$, let $e_A = e_{i_1} \cdots e_{i_k}$. For ($\sim$)$'$ we must show that $e_A t_{ij} \approx e_A$ for any $A \in \mathcal{P}$, and any $i, j \in A$ with $i < j$. Now if A is empty, or if $|A| = 1$, then there is nothing to prove. So suppose that $|A| \geq 2$ and that $i, j \in A$ with $i < j$. Again, choose $\pi \in \mathcal{S}$ with $i = (n-1)\pi$ and $j = n\pi$, and let $w \in \pi\beta^{-1}$. Notice that $\pi^{-1}(n-1, n)\pi = (i, j)$ so that $w^{-1} s_{n-1} w \approx t_{ij}$ since $\ker(\beta) \subseteq \approx$. Then by (P1)$'$, (P2)$'$, (S1), (I3), and Corollary 16, we have

$$\begin{aligned} e_A t_{ij} \approx e_A e_i e_j t_{ij} &\approx e_A w^{-1} e_{n-1} e_n s_{n-1} w = e_A w^{-1} s_{n-1} e s_{n-1} e s_{n-1} w \\ &\approx e_A w^{-1} s_{n-1} e s_{n-1} e w = e_A w^{-1} e_{n-1} e_n w \approx e_A e_i e_j \approx e_A. \end{aligned}$$

This completes the proof. □

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